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Generalised spectra for the dimensions of strange sets

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Abstract. We give a thermodynamical description of the spectrum of generalised dimensions for invariant sets recently proposed by Tél. Our approach allows us to define an uncountable set of generalised spectra, depending on the equilibrium measure supported by the set, and we indicate that the dimensions are independent of the measurable partition of the set.

1. Introduction

In a recent paper [1], Tél has introduced a new spectrum of generalised dimensions to characterise the multifractal nature [2] of the dynamical invariant sets. In this paper we give a new different interpretation of that spectrum, which leads us also to define an uncountable set of generalised spectra, each one depending on the equilibrium measure (Gibbs measure) supported by the set. Until now and for hyperbolic sets, the generalised dimensions have been referred to two equilibrium measures, the maximal entropy measure [3, 4] and the physical measure (Sinai-Bowen-Ruelle (SBR) measure) [1, 5]. As a consequence of our results, we will see that the generalised dimensions computed with respect to the SBR measure coincide with the spectrum introduced by Tèl thus giving a positive answer to the related conjecture formulated in [1].

Before proceeding, it is important to point out to what dynamical systems our considerations apply: we consider here the conformal mixing repellers J of transformations T of the class C^2 [6]. Among these systems, there is a non-trivial class which is particularly simple to study and which exhibits, up to some technicalities, the same properties as the general case. It consists of the uniformly expanding maps T of the unit interval $[0, 1]$, such that T^{-1} $[0, 1]$ consists of s disjoint intervals. If we put A_k^n, $k=1, \ldots, s^n$, the disjoint sets of the collection T^{-n} $[0, 1]$, then the invariant set J is the Cantor set:

$$J=\bigcap_{n=1}^{\infty} \bigcup_{k=1}^{s^n} A_k^n. \tag{1}$$

We call these systems expanding maps of the interval (EMI) [7, 8].

Now we define the various spectra of the generalised dimensions: J will always denote the invariant set, and we put on it a probability measure μ. Following [3, 9], we cover J with a μ-measurable partition $\mathscr{A}^0$ such that diam $\mathscr{A}^n \to 0$, $n \to +\infty$ (generating partition), where $\mathscr{A}^n = \bigvee_{i=0}^{n} T^{-i}\mathscr{A}^0$ is the dynamical partition. Then we introduce the partition function, for real q and τ $(A_k^n \in \mathscr{A}^n)$:

$$H_{q,n}^{\tau}(J)=\sum_{A_k^n}\left(\frac{\mu(A_k^n)^q}{|A_k^n|^\tau}\right) \tag{2}$$

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where $|A| \equiv$ diameter of A. If it happens that, for fixed q and when $n \to +\infty$, there is a change-over point $\tau_{q,\mu}$ such that $H^{\tau}_{q,n}(J)$ goes to zero for $\tau < \tau_{q,\mu}$ and tends to infinity for $\tau > \tau_{q,\mu}$, we put

$$\tau_{q,\mu} = D_{q,\mu}(J)(q-1) \tag{3}$$

and call $D_{q,\mu}(J)$ the generalised dimension of order q of the set J. We emphasise the fact that these dimensions depend on the measure μ and, in general, on the partition $\mathscr{A}^0$. For mixing repellers J it is useful to consider a Markov partition of J [10]; for the EMI every partition $\mathscr{A}^n$, $n > 0$, of the type used in (1) is a Markov partition with disjoint rectangles $A^n_k \cap J$.

There is a probability measure, supported on J, which is of special importance to compute the spectrum $D_{q,\mu}(J)$. It is the measure of maximal entropy, denoted by μ_b which is, obviously, T invariant and ergodic and satisfies the balancement property [7, 11]: $\mu_b(TA) = s\mu_b(A)$, for every measurable set $A \subset J$ where T is injective. In [4, 7, 9] it has been shown that if a conformal mixing repeller J is Markov partitoned and is endowed with the balanced measure μ_b, the relative change-over point τ_{q,μ_b} satisfies the equation

$$P(-\tau_{q,\mu_b}) = qP(0) = qh_{\mathrm{TOP}} = q \log s \tag{4}$$

where h_{TOP} is the topological entropy and $P(\beta)$ is the topological pressure of the function $-\beta \log\|T'(x)\|$ [10], i.e.

$$P(\beta) = \sup_{\mu \in M_T(J)} \left\{ h(\mu) - \beta \int_J \log\|T'(x)\| \, \mathrm{d}\mu(x) \right\} \qquad \beta \in \mathbb{R} \tag{5}$$

$M_T(J)$ being the set of the T-invariant probability measures on J, $h(\mu)$ the μ-Komogorov entropy and $\|T'(x)\|$ the norm of the tangent mapping at x: if the latter is a scalar times an isometry, the mixing repeller is said to be conformal (for example, Julia sets).

For the systems we are dealing with and for every real β, there is only one ergodic measure $\mu_\beta \in M_T(J)$ such that $P(\beta) = h(\mu_\beta) - \beta \int_J \log\|T'(x)\| \, \mathrm{d}\mu_\beta$, and this measure is called the (Gibbs) equilibrium measure for the function $-\beta \log\|T'(x)\|$. For conformal repellers, the integral $\int_J \log\|T'(x)\| \, \mathrm{d}\mu_\beta$ is just the unique μ_β Lyapunov exponent, $\Lambda(\mu_\beta)$, so that (5) can be simply written as

$$P(\beta) = h(\mu_\beta) - \beta\Lambda(\mu_\beta). \tag{5'}$$

Three equilibrium measures have been considered in particular. When $\beta = 0$, the corresponding measure μ_0 is just the balanced measure μ_b; when $\beta = d_{\mathrm{H}} \equiv$ Hausdorff dimension of J, the equilibrium measure $\mu_{d_{\mathrm{H}}}$ satisfies the well known Bowen–Ruelle formula [6]: $P(d_{\mathrm{H}}) = h(\mu_{d_{\mathrm{H}}}) - d_{\mathrm{H}}\Lambda(\mu_{d_{\mathrm{H}}}) = 0$, and is equivalent to the d_{H}-Hausdorff measure of J. Finally, if $\beta = 1$ the corresponding measure satisfies† $P(1) = h(\mu_1) - \Lambda(\mu_1) = -\alpha$ [5, 7], where α is the escape rate [12], and for this formal analogy with the physical measure for axiom-A attractors, it has been called the Sinai–Bowen–Ruelle measure μ_{SBR} in [5, 13]. From now on, the generalised dimensions will be indicated as $D^{\beta}_q(J)$, where the index β refers to the equilibrium measure μ_β put in the partion function (2).

Now we return to the definition of the generalised spectra introduced in [1]. It does not depend, at least in principle, on any measurable partition on the repeller.

† If the tangent mapping is n dimensional, we must put $\beta = n$.

We start by considering the linear Ruelle-Perron-Frobenius operator L_β from the space of continuous functions f on J into itself, of the form

$$L_\beta f(x) = \sum_{y \in T^{-1}x} \frac{f(y)}{\|T'(y)\|^\beta} \qquad x \in J, \beta \in \mathbb{R}. \tag{6}$$

If we put for any real fixed q: $\beta = q - \bar{\tau}$ and $\lambda = \exp(-\alpha q)$, α being the escape rate, then it was argued in [1], using numerical methods, that there exists a unique exponent $\bar{\tau}_q$ for which $\lambda^{-n} L_\beta^n f(x)$ converges uniformly on J to a unique continuous function, for any choice of a smooth f. The numbers

$$\bar{\tau}_q = \bar{D}_q(J)(q-1) \tag{7}$$

define a new spectrum of generalised dimensions. In the case $\beta = d_{\mathrm{H}}$, the eigenvalue problem (6) was previously analysed in [14]; see also [15] for related questions.

2. Dimensions of the Gibbs measure

We now show that the $\bar{D}_q(J)$ are related to the pressure as well. The reason is that the following theorem can be applied to conformal mixing repellers.

Theorem [16]. If T is a C^2 transformation of the conformal mixing repeller J onto itself, there exist, and are unique, a probability Borel measure ν on J, a constant $\lambda > 0$ and a positive function g such that

$$L_\beta^* \nu = \lambda \nu \qquad (L^* \text{ is the dual of } L) \tag{8a}$$

$$L_\beta g = \lambda g \qquad \int_J g \, \mathrm{d}\nu = 1 \tag{8b}$$

$$\lim_{n \to +\infty} \lambda^{-n} L_\beta^n f(x) = g(x) \int_J f \, \mathrm{d}\nu \qquad \text{uniformly on } J \tag{8c}$$

$$\lim_{n \to +\infty} n^{-1} \log L_\beta^n 1 = \log \lambda = P(\beta) \tag{8d}$$

and

$$\mu_\beta = g\nu \tag{8e}$$

i.e. the equilibrium measure for the function $-\beta \log\|T'(x)\|$ is equivalent (with continuous Radon-Nykodim derivative) to ν.

Adapting this theorem to our case, having put $\lambda = \exp(-\alpha q)$, we immediately obtain

$$P(q - \bar{D}_q(J)(q-1)) = -\alpha q. \tag{9}$$

The interpretation of λ in terms of pressure has been given with heuristic arguments in [17] also. The proof of equation (9) is particularly simple for the EMI. In fact, if $\mathscr{A}^0$ is the partition of $[0, 1]$ given by the s sets $T^{-1}[0, 1]$, then every set of the partition $\mathscr{A}^{n-1} = \bigvee_{i=0}^{n-1} T^{-i}\mathscr{A}^0$ contains one and only one point of the set $T^{-n}z$, $z \in J$. By Lagrange's theorem it is easy to see that the diameter of each atom $A_k^{n-1} \in \mathscr{A}^{n-1}$, $k = 1, \ldots, s^n$, is given by

$$|A_k^{n-1}| = \|(T^n)'(\xi)\|^{-1} \tag{10}$$

where ξ is a point in $\mathrm{int}(A_k^{n-1})$. Then, by a well known distorsion argument (see, for example, [18]), given two points x and y in the same A_k^{n-1} we have the uniform bounds

$$C^{-1} \leqslant \frac{\|(T^n)'(x)\|}{\|(T^n)'(y)\|} \leqslant C \tag{11}$$

where $C \geqslant 1$ is independent of $n > 0$. Substituting $y \in (T^{-n}z \cap A_k^{n-1})$ with the ξ given by (10) and using (11), we can bound (6) as (we suppose here without restriction that $q - \bar{\tau} > 0$)

$$\frac{C^{-(q-\bar{\tau})}}{\exp(-\alpha qn)} \sum \frac{1}{|A_k^{n-1}|^{q-\bar{\tau}}} \leqslant \frac{1}{\exp(-\alpha qn)} L_{q-\bar{\tau}}^n 1$$
$$\leqslant \frac{C^{q-\bar{\tau}}}{\exp(-\alpha qn)} \sum \frac{1}{|A_k^{n-1}|^{q-\bar{\tau}}} \tag{12}$$

where the sums are over $A_k^{n-1} \in \mathscr{A}^{n-1}$. When $n \to +\infty$, it can be shown that the thermodynamical limit of the partition function $\Sigma\, 1/|A_k^{n-1}|^\beta$ is just the pressure $P(\beta)$ and the convergence is uniform in β on any compact subset of $\mathbb{R}$, [7, 8] (this also gives a good method to compute the pressure numerically [8]; another powerful technique which is valid also for connected repellers follows directly from (8d)). In order for the central term of (12) to converge to a strictly positive function, it is easy to see that equation (9) must be true.

Conditions (8a) and (8d) imply that

$$\nu(TA) = \int_A \frac{\|T'(x)\|^\beta}{\exp[-P(\beta)]}\, \mathrm{d}\nu(x) \tag{13}$$

where A is any measurable subset of J where T is injective. Now, repeating an argument quoted in [9], given $\varepsilon > 0$ we can find a Markov partition $\mathscr{A}^0$ of $[0,1] \cap J$ such that for any two points x, y in an atom of $\mathscr{A}^0$ we have $\|T'(x)\| \leqslant e^\varepsilon \|T'(y)\|$. Then applying iteratively the relation (13) to the elements $A_K^{n-1} \in \mathscr{A}^{n-1} = \bigvee_{K=0}^{n-1} T^{-K}\mathscr{A}^0$, replacing the measure ν (up to a finite constant) by (8e) with the Gibbs measure μ_β and using again the identity (10), we can bound the measure of an atom A_K^{n-1} as (we refer to [9] for the detail; in addition, we suppose without restriction that $\beta > 0$)

$$\mathrm{e}^{n[-\varepsilon - P(\beta)]} |A_k^{n-1}|^\beta \mu_\beta(T^n A_k^{n-1}) \leqslant \mu_\beta(A_k^{n-1})$$
$$\leqslant \mathrm{e}^{-n[-\varepsilon + P(\beta)]} |A_k^{n-1}|^\beta \mu_\beta(T^n A_k^{n-1}) \tag{14}$$

where $\mu_\beta(T^n A_k^{n-1})$ is the finite measure of an element of the partition $\mathscr{A}^0$. The relation (14) is quite useful to compute the generalised dimensions with respect to the same measure μ_β. In fact, substituting it into (2) and again using the nature of the thermodynamical limit of the partition function $\Sigma\, 1/|A_k^{n-1}|^\beta$, we obtain immediately that the change-over point τ_{q,μ_β} is uniquely defined and the generalised dimensions $D_q^\beta(J)$ satisfy

$$P(\beta q - D_q^\beta(J)(q-1)) = qP(\beta). \tag{15}$$

When $\beta = 0$ we recover (4); for $\beta = 1$ we reproduce Tel's spectrum: it is thus obtained by computing the generalised dimensions by the use of the Sinai-Bowen-Ruelle measure. Finally, when $\beta = d_H$ the dimensions themselves are simply equal to the Hausdorff dimension, as already noted in [9]. The bounds (14) allow us also to compute the Renyi entropies $K_q(\mu)$ [19] of the Gibbs measure μ_β. We recall that the Renyi entropies are defined as, the partition $\mathscr{A}^0$ being generating,

$$K_q(\mu)(1-q)=\lim_{n\to\infty}\frac{1}{n}\log\sum_{A_k^n\in\mathscr{A}^n}\mu(A_k^n)^q. \tag{16}$$

Using arguments similar to those which lead to (15) we obtain

$$K_q(\mu_\beta)(1-q)=-qP(\beta)+P(q\beta). \tag{17}$$

For $\beta=1$ and $\beta=d_{\mathrm{H}}$ we recover respectively the formulae already quoted in [5] and [9]. Finally we note that (15) and (17) also hold for general conformal mixing repellers if the techniques of theorem 4.6 in [7] are used.

We now come back to (15). All these equations give, for $q=0$, that $D_0^\beta=d_{\mathrm{H}}=$ Hausdorff dimension of the set, since the pressure is strictly monotone and $P(d_{\mathrm{H}})=0$ by the Bowen–Ruelle formula.

If the mapping T is uniformly expanding on J, there is a power r of T for which surely $\|(T^r)'(x)\|\geqslant 1$, $x\in J$. Using T^r instead of T (or alternatively an 'adapted' metric on J) and the properties (v) and (ix) of the pressure quoted in theorem 2.1 in [20], it is easy to show that all the $D_q^\beta(J)$ are non-negative, so they are well defined as dimensions. Moreover, since the pressure $P(\eta)$ is real analytic in η and deriving (15), we obtain

$$D_1^\beta(J)=\beta-P(\beta)\left(\frac{\partial P(\eta)}{\partial\eta}\bigg|_{\eta=\beta}\right)^{-1}. \tag{18}$$

The unexpected fact is that the $D_q^\beta(J)$ for certain q and β can be greater than the topological dimension of the ambient space. Let us consider, for example, a linear EMI with two scales γ_1 and γ_2, $\gamma_1+\gamma_2<1$; in this case the pressure has the expression

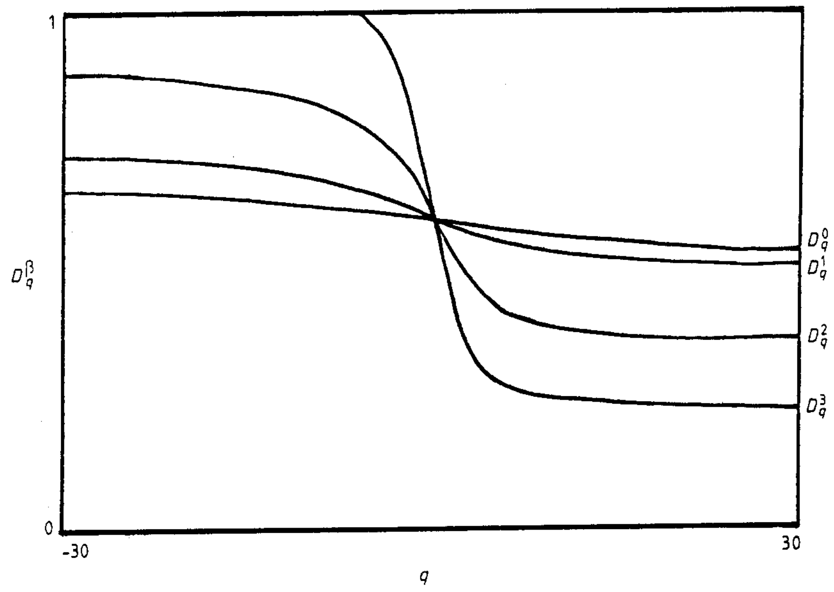

Figure 1. The dimensions D_q^β against q for $\beta=0, 1, 2, 3$ for the Cantor set with two scales $\gamma_1=0.25$, $\gamma_2=0.4$. They have been computed by solving $\gamma_1^{\beta q-D_q^\beta(q-1)}+\gamma_2^{\beta q-D_q^\beta(q-1)}=(\gamma_1^\beta+\gamma_2^\beta)^q$ using a Newton method. The Hausdorff dimension is $d_{\mathrm{H}}=0.610\,98$.

[7]: $P(\eta)=\log(\gamma_1^\eta+\gamma_2^\eta)$. If $\gamma_1=\gamma_2$, the D_q^β for fixed q and β variable are all equal (homogenuous fractal). If instead we take, say, $\gamma_1=0.25$ and $\gamma_2=0.4$, and compute (18) by means of the analytic expression of the pressure, we get $D_1^0=0.602$, $D_1^1=0.607$, $D_1^2=0.566$ and $D_1^3=0.491$. In figure 1 we sketch D_q^β against q for $\beta=0$, 1, 2, 3; as we see, D_q^3 for $q\leqslant -5$ becomes greater than one. Tél [21] has communicated to the author that he also observed a similar behaviour for D_q^1 in another Cantorian system.

3. Hausdorff measures

Now we return to (8) and put $\beta\equiv$ exponent of $\|T'(x)\|=\eta$, to avoid future confusion with the subscript β for the Gibbs measure.

If we put $\eta=d_{\mathrm{H}}$ it is known that the constant λ given by (8a) is equal to one and the corresponding 'conformal' measure ν is equivalent to the d_{H}-Hausdorff measure of J [6]. If instead we put, for fixed q, $\eta=q\beta-\tau_{q,\mu_\beta}$, τ_{q,μ_β} being the change-over point defined in (3), we have seen that λ becomes equal to $\exp(qP(\beta))$; it is therefore natural to ask what the meaning of the corresponding measure ν is in terms of the local fractal properties of J.

To give an answer to this question, we have to change once more the definition of the generalised dimensions given through (2), in such a way as to make them independent of the partition of the invariant set J. Following [7] closely, we modify (2) as

$$H_{q,l}^\tau(J)=\inf_{B_k(x)\in\beta_l}\sum_{k=1}^{\infty}\frac{\mu(B_k(x))^q}{|B_k(x)|^\tau} \tag{19}$$

where the infimum is over all countable coverings $\beta_l\equiv\{B_k(x)\}_{k=1}^\infty$ of J by closed balls $B_k(x)$ of centre $x\in J$ and of diameter less than l. It can be shown that $H_q^\tau(J)=\lim_{l\to 0}H_{q,l}^\tau(J)$ is a Borel measure on J and that, for any fixed q, there exists a change-over point $\hat\tau_q(J)$ such that $H_q^\tau(J)$ is zero when $\tau<\hat\tau_q(J)$ and is infinite when $\tau>\hat\tau_q(J)$, and $\hat\tau_q(\bar J)\geqslant\hat\tau_q(J)$ when $\bar J\subset J$. The generalised dimensions are then defined in the standard way: $\hat D_q(J)(q-1)=\hat\tau_q(J)$. If we refer the $\hat D_q(J)$ to the subsets of full μ-measure and use the monotony of $\hat\tau_q(J)$ as a set function, we are led to define the generalised dimensions of the measure as

$$mD_q(\mu)=\inf_{\substack{Y\subset J\\ \mu(Y)=1}}\hat D_q(Y)\qquad\text{when } q\leqslant 1 \tag{20a}$$

and

$$MD_q(\mu)=\sup_{\substack{Y\subset J\\ \mu(Y)=1}}\hat D_q(Y)\qquad\text{when } q\geqslant 1. \tag{20b}$$

For conformal mixing repellers (and other systems) we have proved in [7] that $mD_q(\mu)=MD_q(\mu)=\mathrm{HD}(\mu)$, $\forall q\in\mathbb{R}$, $\mathrm{HD}(\mu)$ being the Hausdorff dimension of the measure μ, i.e.

$$\mathrm{HD}(\mu)=\inf_{\substack{Y\subset J\\ \mu(Y)=1}}\{\text{Hausdorff dimension of } Y\}.$$

As a consequence of (20) we have that $\hat D_q(J)\leqslant\mathrm{HD}(\mu)$, $\forall q\geqslant 1$ and $\hat D_q(J)\geqslant\mathrm{HD}(\mu)$, $\forall q\leqslant 1$. These inequalities agree, in the light of the following claim applied to the balanced measure for which $\hat D_q(J)=D_q^0(J)$, with the behaviour of $D_q^0(J)$, shown in figure 1. In fact, for this system we have [7] $\mathrm{HD}(\mu_b) = h(\mu_b)/\Lambda(\mu_b)$, $\Lambda(\mu_b)$ being the Lyapunov exponent of the maximal entropy measure. But $h(\mu_b) = P(0)$ and $\Lambda(\mu_b) = -\partial P(\beta)/\partial\beta|_{\beta=0}$ [10], and so, by (18), $\mathrm{HD}(\mu_b) = D_1^0$. Coming back to the interpretation of the measure ν defined by the eigenvalue equation $L^*_{(q\beta-\tau_{q,\mu_\beta})}\nu = \exp(qP(\beta))\nu$, we are now in the position to formulate the following.

Conjecture. If we put on the repeller the equilibrium measure μ_β, the measure ν defined by the above eigenvalue equation is equivalent to the $H_q^{\tau_{q,\mu_\beta}}(J)$-measure, where τ_{q,μ_β} is defined by (15).

As a consequence, we have that the change-over point τ_{q,μ_β} is equal to the corresponding one $\hat{\tau}_q$ defined via (19) having put the measure μ_β on the set. Then the generalised spectra $\hat{D}_q(J)$ introduced in a 'partition-independent' way will satisfy the equation $P(\beta q - \hat{D}_q(J)(q-1)) = qP(\beta)$, which generalises the similar 'partition-dependent' relation (15). We see that, for $q=0$, $-\hat{\tau}_0 = \hat{D}_0(J) = d_H$ and $H_0^{\hat{\tau}_0}(J)$ is just the d_H-Hausdorff measure; moreover we recover the Bowen-Ruelle formula $P(d_H) = 0$.

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Note added. After this paper was submitted, we became aware of a paper by Bohr and Tél [22] in which results similar to ours have been obtained.

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